Motives and Milnor conjecture

notes by S. Kelly:

Day III

Contents

1 S. Gille, Chow motives and motives of quadrics I

We will not enter much into the philosophy of motives, but go directly to the definition of Rost motives. We consider F a field of characteristic different from 2, PSm_F = the category of smooth projective F-schemes, and $CH_\bullet$ the Chow group of dimension $\bullet$-cycles modulo rational equivalence. The product $X \times Y$ is always over F, and for a field extension E=F, we define $X_E = X \times E$ and ${}_E = \times E$ for $\in CH_\bullet(X)$.

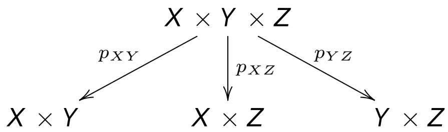
1.1 Construction of Chow motives

The category of Chow motives can be thought of roughly as a "linearisation" of PSm_F. Here Chow motives will always be with integral coefficients.

The first step is to "make" PSh_F additive.

Definition. We define $Corr^0(F)$ the *category of correpsondences of degree zero.* The objects of this category are the objects of PSh_F but $\hom_{Corr^0(F)}(X; Y) = \oplus_{i=1}^{\ell} CH_{\dim\ X_i}(X_i \times Y)$ where X_i are the connected components of X. Elements of $\hom_{Corr^0(F)}(X; Y)$ are called correspondences of degree zero and we use the notation $\ : X \vdash Y$.

Composition is defined as follows. Suppose that X and Y are irreducible. For $\ : X \vdash Y$ and $\ : Y \vdash Z$ we have a diagram of projections



and we define

$$\circ \quad = (p_{XZ})_* \left(p_{XY}^* \quad \cap p_{YZ}^* \quad \right):$$

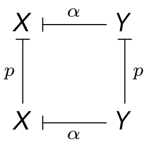
Exercise 1. With this composition $Corr^0(F)$ is an additive category, the identity morphism is the class of the diagonal in $CH(X \times X)$ and the direct sum is $X \oplus Y = X \amalg Y$ where $\amalg$ is the disjoint union.

The second step is to be able to split of idempotents (that is, every endomorphism p that satisfies p^2 should define a direct sum decomposition of the relevant object). We define the category $Chow(F)$ as the idempotent completion of $Corr^0(F)$. Heuristically, this involves formally adding kernels and cokernels of idempotent endomorphisms, i.e. correspondences $p \in \hom_{Corr^0(F)}(X; X)$ such that $p \circ p = p$.

Definition. Formally, we define $Chow(F)$ to be the category whose objects are pairs $(X; p)$ where X is an object of $Corr^0(F)$ and $p \in \hom_{Corr^0(F)}(X; X)$ such that $p \circ p = p$. The morphisms are

$$\hom_{Chow(F)}((X; p); (Y; q)) = q \circ \hom_{Corr^0(F)}(X; Y) \circ p:$$

This latter is equal to the set of $\ : X \vdash Y$ such that



is a commutative square, modulo the relation that $\ = 0$ if $\ \circ p = q \circ \ = 0$.

Remark 2. This is an additive category with the property that if $p \in \hom_{Corr^0(F)}(X; X)$ is an idempotent endomorphism then in $Chow(F)$ we have $X \cong (X; p) \oplus (X; id_X - p)$. In fact, this is the universal category with this property. There a functor

$$PSm_F \to Chow(F)$$

which sends X to $(X; id)$ where the latter will usually be denoted as X. A morphism $f : X \to Y$ is send to the class of the graph $[\Gamma_f] \in Ch(X \times Y)$.

Exercise 3. This is a functor.

Remark 4.

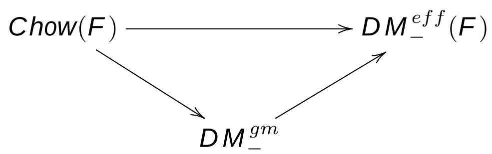
1. $Chow(F)$ is a tensor category with the definition $(X; p) \otimes (Y; q) = (X \times Y; p \times q)$.

2. This category has no more information than the Chow ring. For example, there is a one-to-one correspondence between decompositions of X in $Chow(X)$ and idempotents in $Chow_{\dim\ X}(X \times X)$.

3. Chow motives can be defined with coefficients in a ring R, which is denoted $Chow(F; R)$. We use the same construction but replace $CH_\bullet$ with $R \times_{\mathbb{Z}} CH_\bullet$.

4. (Yogita/Vishik) If $Char\ F = 0$ and one replaces $CH_\bullet$ by a universal cohomology theory, for example $\Omega_\bullet$ = algebraic cobordism, we get the same decomposition result.

5. There are functors

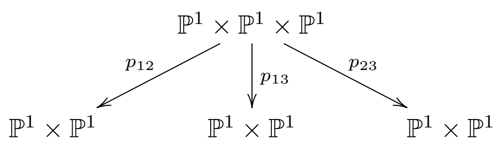


which are fully faithful if the characteristic of F is zero.

Example 5.

0. $Spec\ F$. We have $End(Spec\ F) = CH_0(F) = \mathbb{Z}$ and so $Spec\ F$ is irreducible. Notation: $\mathbb{1} = Spec\ F$ (this is the unit for the tensor product structure on $Chow(F)$).

1. $\mathbb{P}^1_F$. The class of the diagonal in $Ch_1(\mathbb{P}^1 \times \mathbb{P}^1)$ is equal to $[\mathbb{P}^1 \times pt] + [pt \times \mathbb{P}^1]$ for any choice of rational point pt. The correspondences $[\mathbb{P}^1 \times pt]; [pt \times \mathbb{P}^1]$ easily seen to be orthogonal idempotents.



It follows that we get a decomposition

$$\mathbb{P}^1 = \mathbb{1} \oplus \mathbb{N}$$

where $\mathbb{1} = (\mathbb{P}^1; [\mathbb{P}^1 \times pt])$ and $\mathbb{N} = (\mathbb{P}^1; [pt \times \mathbb{P}^1])$. The latter is called the Leftschetz or Tate motive.

Remark 6.

1. We should think of this decomposition of $\mathbb{P}^1$ as a decomposition as a point and an affine line.

2. What we are calling Chow motives are usually called effective Chow motives and what are usually called Chow motives are obtained by formally adjoining a tensor inverse to $\mathbb{N}$ to the category of effective Chow motives.

3. For an object M of $Chow(F)$ we define $M(i) = M \otimes \mathbb{N}^i$.

Exercise 7. If X is irreducible, then

$$\hom_{Chow(F)}(X(i); Y(j)) = CH_{\dim\ X-i-j}(X \times Y):$$

In particular, $\hom_{Chow(F)}(\mathbb{N}^i; \mathbb{N}^j) = \begin{cases} 0 & i \neq j \\ \mathbb{Z} & i = j \end{cases}$.

Example 8.

1. We have a decomposition $\mathbb{P}^n \cong \mathbb{1} \oplus \mathbb{N} \oplus \mathbb{N}^2 \oplus \cdots \oplus \mathbb{N}^n$.

2. Also, in general for any rational point of a variety $x \in X$ we have the idempotent endomorphisms $[X \times x]; [x \times X]$ which give a decomposition $\mathbb{1} \oplus M \oplus \mathbb{N}^{\dim\ X}$.

1.2 Decomposition of motives of quadrics

Let $q = \sum_{i=0}^n a_i T_i^2$ be a regular quadratic form with $a_i \in F\backslash\{0\}$. We define $X_q \subset \mathbb{P}^n$ the corresponding variety which is a smooth projective quadric which is irreducible if $n \geq 2$. If q is isotropic (equivalently, there exists a rational point of X) there is a filtration of X_q. We have $q \cong S \cdot T + \sum_{i=2}^n b_i T_i^2$ for some $b_i \in F\backslash\{0\}$. We have $X_q \backslash (S = 0) \cong \mathbb{A}^{n-1}$ and there is a $\mathbb{A}^1$-bundle $(S = 0)\backslash\{[0 : 1 : \cdots : 0] \to X_{q'}$ where $q' = \sum_{i=2}^n b_i T_i^2$ and $[0 : b : x_2 : \cdots : x_n] \mapsto [x_2 : \cdots : x_n]$.

Theorem 9 (Rost)**.** *If q is isotropic, q' as above, then in $Chow(F)$ we have $X_q \cong \mathbb{N}(\dim X_q) \oplus X_{q'}(1) \oplus \mathbb{1}$.*

Remark 10. Consequently, if q is split i.e.

$$q \cong \begin{cases} \sum_{i=0}^m S_i T_i & \text{if dim } X_q \text{ is even} \\ \sum_{i=0}^m S_i T_i + aT^2 & \text{if dim } X_q \text{ is odd } ; a \in T\backslash\{c\} \end{cases}$$

then

$$X_q \cong \begin{cases} \mathbb{1} \oplus \mathbb{N} \oplus \cdots \oplus \mathbb{N}(\frac{\dim\ X_q}{2})^{\oplus 2} \oplus \cdots \oplus \mathbb{N}(\dim\ X_q) \\ \mathbb{1} \oplus \mathbb{N} \oplus \cdots \oplus \mathbb{N}(\dim\ X_q) \end{cases}$$

in $Chow(F)$. If we use coefficients R with $\frac{1}{2} \in R$ then this is true for non-split quadrics as well.

Theorem 11 (Karpenko)**.** *Let* $X \in PSm_F$*. Assume that there are closed subschemes* $X = X_\ell \supset X_{\ell-1} \supset \cdots \supset X_{-1} = \varnothing$ *and we have* $\mathbb{A}^{n_i}$ *bundles* $p_i : X_i \backslash X_{i-1} \to Y_i$ *with* $Y_i \in PSm_F$*. Then*

$$X \cong \oplus_{i=1}^{\ell} Y(n_i):$$

Idea of proof. We use the Manin identity principle (which is actually, no more than Yoneda). It is enough to show that $\hom_{Chow(F)}(M; \oplus_{i=1}^{\ell} Y(n_i)) \cong \hom_{Chow(F)}(M; X)$ for all $M \in Chow(F)$. Since they are direct summands of some $Z \in PSm_F$ it is enough to show it for $Z = M$. Recall that for $\in CH_\bullet(X \times Y)$ we define ${}^t \in CH_\bullet(Y \times X)$ as pullback along the isomorphism $X \times Y \cong Y \times X$. How do we construct this isomorphism

$$\hom(Z; \oplus Y(n_i)) \to \oplus Ch_{\dim\ Z - n_i}(Z \times Y) \to Ch_{\dim\ Z}(Z \times X)?$$

Let $_i$ = the closure — of graph of $p_i : X_i \backslash X_{i-1} \to Y_i$ in $X_i \times Y_i$. Define $_i = \Gamma_{f_i} \circ {}_i^t$ where the former is the graph of $f_i : X_i \to X$. We then show that

$$\sum_{i=0}^{\ell} (\Gamma_{f_i} \circ {}_i^t)_* : \oplus Ch_{\bullet - n_i}(Z \times Y_i) \to Ch_\bullet(Z \times X)$$

is an isomorphism. □

2 S. Gille, Chow motives and motives of quadrics II

Definition. A motive M is called geometrically split if $M_{\overline{F}} \cong \oplus_{finite} \mathbb{N}(i)^{n_i}$ where $\overline{F}$ is the algebraic closure of F.

Remark 12. Note that X is geometrically split if and only if

$$\oplus_{i+j=d} CH_i(\overline{X}) \otimes CH_j(\overline{X}) \to CH_d(\overline{X} \times \overline{X})$$

is an isomorphism for all d where $\overline{X} = X_{\overline{F}}$.

Example 13. Some varieties that are geometricall split are the following.

1. Projective homogeneous varieties = $G=P$ where G is split semisimple group and P a parabolic subgroup.
2. Toric varieties.

Remark 14. For such varieties you know quite often (like for quadrics) everything about the Chow theory over the closure. Question: Can one use this knowledge to understand the motive over the vase field? Some answer is given by the Rost nilpotence principle (RN).

Definition. One says that Rost nilpotence is true for $X \in PSm_F$ in $Chow(F)$ if: given $\in End(X)$ with the property that $_E = 0$ for some field extension $E{=}F$ then $^{\circ N} = 0$ for some $N > 0$. In other words, the kernel of $(-)_E : End_F(X) \to End_E(X)$ consists of nilpotent elements.

We will see a proof that this is true for quadrics but first we show some consequences.

Remark 15. In $Chow(F;\mathbb{Q})$ this is always true because the morphism $End_F(X)\otimes \mathbb{Q} \to End_E(X) \otimes \mathbb{Q}$ is always an injection.

Some consequences of RN. Assume that RN is true for $X \in Chow(F)$.

1. If $p \in End_F(X)$ is idempotent and nonzero, then p_E is nonzero for every field extension $E{=}F$.
2. If $p \in End_F(X)$ becomes an idempotent over some extension field $E{=}F$ i.e. $p_E \circ p_E = p_E$ then there exists $\tilde{p} \in End_F(X)$ idempotent with $\tilde{p}_E = p_E$.
3. If $X;Y$ are geometrically split and RN holds in $Chow(F)$, and $: X \vdash Y$ is an isomorphism over some field extension $E{=}F$ then is already an isomorphism.

Proof. By enlarging F we can assume that $X_E;Y_E$ are split.

1. Case $X = Y$: Then $_E \in Aut_E(X_E) \cong \prod_{n_i} GL_{n_i}\mathbb{Z}$ if $X_E \cong \oplus \mathbb{N}^{n_i}$. Let f = characteristic polynomial of $_E$. Then $f(\ _E) = 0$, but $f(0) = \det\ _E = \pm 1$. This means that $f = Tg(T) \pm 1$ and so $0 = f(\ _E) = \ _E g(\ _E) \pm 1$. It follows that $_E(\pm g(\ _E)) = id$ and Rost nilpotence then says that $(\pm g(\ )) = id+$ nilpotent, which is an isomorphism. The same holds for $\pm g(\ )$ and so is an isomorphism.
2. General case: $_{E*} : CH_i(X_E) \to CH_i(Y_E)$ (where $CH_i(X_E) = \hom(\mathbb{N}^i; X_E)$ etc and this morphism is composition) is an isomorphism since $_E$ is an isomorphism and $\dim X = \dim Y$. This implies that ^{t} is a correspondence of degree zero $Y \vdash X$. Then $\circ\ ^t : Y \vdash Y$ and $^t \circ\ : X \vdash X$ are correspondences of degree zero which are isomorphisms over E. The result follows from the previous case then.

□

Remark 16. If R = finite field, then if RN is true for X in $Chow(F;R)$ and X is geometrically split, then Vivult-Schmidt is true for X in $Chow(F;R)$ (there is a decomposition into indecomposable motives which is unique up to isomorphism).

Theorem 17 (Rost)**.** *RN is true for quadrics in $Chow(F)$ (and also $Chow(F;R)$ for any R).*

The proof uses a lemma of Rost.

Lemma 18. *Let* $\in End_F(X)$ *which* $X \in PSm_F$. *Assume that* $({}_{F(x)})_*(CH_{codim\overline{\{x\}}}(X_{F(x)})) = 0$ *for all* $x \in X$. *Then* ${}^{\circ(\dim X+1)} = 0$.

Proof of the lemma after Brosnan (cf. [Bro03]). Assume X is irreducible. Define a filtration on $CH_{\dim X}(X \times X) : F_p(CH_{\dim X}(X \times X)) = \langle [V] s.t. V \subset X \times X$ is irreducible, $\dim pr_* V \leq p\rangle$ where $pr : X \times X \to X$ is projection to the first component. It is enough to show that ${}_* F_p \subseteq F_{p-1}$. For this it is enough to show that if V is irreducible in $X \times X$ such that $\dim prV = p$ and we define $W = prV$ then ${}_*[V] \in F_{p-1}$. We have a diagram

$$\begin{array}{ccccccc}
 & & CH_d(W \times X) & \to & CH_d(U \times X) & \to & CH^d(F(W) \times X) \to 0 \\
 & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha|_{U*}} & & \downarrow{\scriptstyle \alpha_*[V]=0} \\
CH_d(W \backslash U \times X) & \to & CH_d(W \times X) & \to & CH^d(U \times X) & \to & CH^d(F(W) \times X)
\end{array}$$

□

Proof of RN for the quadric X_q *where* $q = \sum_{i=0}^n a_i T_i^2$. We use induction on $n \geq 1$. We claim that if ${}_E = 0$ for some E/F with $\in End_F(X_q)$ then ${}^{\circ(\dim X_q+1)!} = 0$. If $n = 1$ then X_q is either a quadratic field extension or two rational points in which case it is clear. If $n = 2$ then X_q is a form of $\mathbb{P}^1$ and $End_F X_q = Pic(X_q \times X_q)$ and it follows from the Hoschild-Serre spectral sequence.

Let $x \in X_q$. Then $X_q \times F(x)$ is an isotropic quadric. The Rost decomposition theorem implies that $X_q \times F(x) = \mathbb{1} \oplus X_{q'}(1) \oplus \mathbb{N}^{\dim X_q}$ in $Chow(F(x))$ where q' is a quadratic form over $F(x)$ with $\dim X_{q'} = \dim X_q - 2$. We have

$$End_{F(x)}(X_q \times F(x)) \cong \mathbb{Z} \times End_{F(x)}(X_{q'}) \times \mathbb{Z}$$

and so ${}_{F(x)} = (0; \ ; 0)$ with ${}_{E(x)} = 0$. Using induction, ${}^{\circ(\dim X_q-1)!} = {}^{\circ(\dim X_{q'}+1)!} = 0$ and so

$${}^{\circ(\dim X_q-1)!} CH_i(X_q \times F(x)) = 0$$

for all $i \in \{0; \ldots; \dim X_q\}$ and all $x \in X$. By Rost's lemma,

$$({}^{\circ(\dim X_q-1)!})^{\circ(\dim X_q+1)} = 0$$

and the claim follows. □

Remark 19.

1. RN is true for projective homogeneous varieties in $Chow(F; R)$ for any coefficient ring R.
2. RN is true for surfaces in $Chow(F; \mathbb{Z})$ if char $F = 0$ and is true for geometric rational surfaces in arbitrary characteristic if R is a finite field, $R = \mathbb{Z}$ or $R = \mathbb{Z}/m$.

3 J. Ayoub, Motivic complexes and motivic cohomology IV

The outline is as follows.

1st lecture:

1. Recalling some facts from Bruno's talk.
2. The cancellation theorem.
3. Relation between motivic cohomology and Chow groups.
4. Duality, embedding of Chow motives
5. Projective bundle formula.

2nd lecture:

1. Relation with Milnor K-theory.
2. Relation with étale cohomology (étale motives, Lichtenbaum motivic cohomology).

References: [BV08] (part I and III of this series of lectures), [MVW06].

Let k be a perfect field, and $Sm{=}k$ the category of smooth varieties. We denote $SmCorr(k)$ the category of smooth varieties with finite correspondences. We have PSh_{tr} the category of presheaves with transfers (PST) i.e. the category of additive functors from $SmCorr(k)$ to abelian groups. Of great importance are the homotopy invariant presheaves with transfers.

Theorem 20 (Voevodsky)**.** *If F is a homotopy invariant presheaf with transfers then $H^*_{Nis}(-;F) \cong H^*_{Zar}(-;F)$ is again a homotopy invariant presheaf with transfers.*

Remark 21.

1. The fact that $H^*_{Nis}(-;F)$ has transfers comes easily from the compatibility of the Nisnevich topology with transfers but the isomorphism with the Zariski cohomology and homotopy invariance is quite hard.
2. If F is as above, then F restricted to the small Zariski site of $\mathbb{A}^1_{k(X)}$ is already a sheaf which is acyclic.
3. If we take a morphism $F \to G$ between homotopy invariant presheaves with transfers such that $F(\ _X) \cong G(\ _X)$ for all $X \in Sm{=}k$ irreducible then $F_{Nis} \cong G_{Nis}$.

Definition. Recall that for F a presheaf with transfers $F_{-1} = \underline{hom}((\mathbb{G}_m;1);F)$

Proposition 22. *Assume that F is homotopy invariant. Then $(F_{Nis})_{-1} \cong (F_{-1})_{Nis}$. More generally, If $K_\bullet \in Comp(PSh_{tr})$ such that $H_\bullet(K)$ is homotopy invariant, then $R(-)_{-1}(K) \cong K_{-1}$.*

Recall that $(-)_{-1}$ is used in the construction of the Cousin complex

$$(F|_{Et/X})_{Nis} \to \prod_{x \in X^{(0)}} F(x) \to \prod_{x \in X^{(1)}} F_{-1}(x) \to \cdots$$

3.1 Cancellation theorem (Voevodsky)

$$DM^{eff}(k) = D(PSh_{tr}) = \left\{ \begin{array}{c} \text{homotopy} \\ \text{Nis top} \end{array} \right\}$$

If $X \in Sm/k$, $M(X) \cong C_\bullet^{\mathbb{A}^1}\mathbb{Z}_{tr}(X) = \mathbb{Z}_{tr}(X)[-]$ is an object of $DM^{eff}(k)$. We have the Leftschetz motive $\mathbb{Z}(1) = \mathbb{Z}_{tr}((\mathbb{G}_m, 1))[-1]$. The motive $\mathbb{Z}(1)$ is not invertible in $DM^{eff}(k)$ but we have the following.

Theorem 23 (Voevodsky)**.** *The endofunctor* $- \otimes \mathbb{Z}(1)$ *on* $DM^{eff}(k)$ *is fully faithful.*

Remark 24.

1. This is equivalent to asking that for every smooth scheme X we have an isomorphism $M(X) \cong \underline{hom}(\mathbb{Z}(1), X \otimes \mathbb{Z}(1))$. This would follow from $C_\bullet\mathbb{Z}_{tr}(X) \to RHom((\mathbb{G}_m, 1), C_\bullet\mathbb{Z}_{tr}(X \times \mathbb{G}_m))$ being an $\mathbb{A}^1$ weak equivalence. There is no need for $RHom$ in fact, we can just take hom.

2. Voevodsky constructs a partially defined retraction up to homotopy. For a divisor $Z \subset \mathbb{G}_m \times \mathbb{G}_m \times X$ (i.e. an element on the right) we intersect with $Div(\frac{t_1^n - t_2}{t_1^n - 1})$.

3.2 Relation between motivic cohomology and Chow groups

The motivic complexes are

$$\mathbb{Z}(n) = C_\bullet\mathbb{Z}_{tr}((\mathbb{G}_m, 1)^{\wedge n})[-n]$$

and motivic cohomology is

$$H_M^{p,q}(X) = \mathbb{H}^p_{Nis}(X, \mathbb{Z}(q)).$$

We have the following comparison result with Chow groups.

Theorem 25.

$$H^{2m,m}(X) \cong CH(X).$$

Note that if we contract $\mathbb{Z}(n)$ we get $\mathbb{Z}(n)_{-1} = \mathbb{Z}(n-1)[-1]$. Iterating we get $\mathbb{Z}(n)_{-n} = \mathbb{Z}[-n]$ and $\mathbb{Z}(n)_{-n+1} = \mathcal{O}^*[-n]$. For $p, q \in \mathbb{Z}$ we denote $\mathcal{H}^{p,q} =$ the sheaf associated to motivic cohomology $H_M^{p,q}(-)$. We see that $\mathcal{H}^{n,n}(-)$ is the first non-zaro homology sheaf of $\mathbb{Z}(n)$.

From the cancellation theorem we can restate this isomorphism as

$$(\mathcal{H}^{p,q})_{-1} = \mathcal{H}^{p-1,q-1}$$
$$(\mathcal{H}^{p,n})_{-n} = \mathbb{Z} \text{ for } p = n$$
$$(\mathcal{H}^{p,n})_{-n} = 0 \text{ for } p \neq n$$
$$(\mathcal{H}^{p,q})_{-n+1} = 0 \text{ for } p \neq n$$
$$(\mathcal{H}^{p,q})_{-n+1} = \mathcal{O}^* \text{ for } p = n$$

Now for $X \in Sm=k$ $H^\bullet(X; \mathcal{H}^{p,q})$ can be computed using the Gersetn (Cousin) resolution

$$\mathcal{H}^{p,n}(\ _x) \to \prod_{x \in X^{(1)}} \mathcal{H}^{p-1,n-1}(k(x)) \to \cdots \to \prod_{x \in X^{(n)}} \mathcal{H}^{p-n,0}(k(x)) \to 0$$

and then by a spectral sequence argument we get

$$H_M^{m,n}(X) \cong H^{m-n}(X; \mathcal{H}^{n,n})$$

for $m \geq 2n - 2 \geq \dim(X) + n$.

$$H_M^{n,n}(X) = H^n(X; \mathcal{H}^{n,n}) = CH^n(X)$$

If in the Gersten resolution we take $p = n$ we get

$$\mathcal{H}^{n,n}(\ _x) \to \prod_{x \in X^{(1)}} \mathcal{H}^{n-1,n-1}(k(x)) \to \cdots \to \prod_{x \in X^{(n-1)}} k(x)^* \to \prod_{x \in X^{(n)}} \mathbb{Z} \to 0$$

3.3 Duality and embedding of Chow motives

Let X be a smooth variety pure of dimension d. We know that we have $H^{2d,d}(X \times X) = CH^d(X \times X)$. In this group we have the class of the diagonal. Another way to define this would be $\hom_{DM^{eff}}(X \times X; \mathbb{Z}(d)[2d])$. We get in $DM^{eff}(k)$ a morphism $M(X) \otimes M(X) \to \mathbb{Z}(d)[2d]$ which then gives $D : M(X) \to \underline{hom}(X; \mathbb{Z}(d)[2d])$.

Theorem 26. *If X is proper, then D is an isomorphism.*

Remark 27. The proof of this theorem is quite complicated and Voevodsky uses moving cycles techniques to prove it. It doesn't use resolution of singularities. This is true over any perfect field.

We use this to show that the Chow motives are embedded in $DMeff$. Let $X; Y$ be smooth projective varieties. We want to compute the morphisms between them in $DMeff$. We have

$$\begin{aligned}\hom_{DM^{eff}}(X; Y) &= \hom_{DM^{eff}}(X; \underline{hom}(Y; \mathbb{Z}(d_Y)[2d_Y])) \\ &= \hom_{DM^{eff}}(X \times Y; \mathbb{Z}(d_Y)[2d_Y]) \\ &= CH^{d_Y}(X \times Y) \\ &= \hom_{Chow(k)}(X; Y)\end{aligned}$$

So we have proven the following theorem.

Theorem 28. *There is a fully faithful embedding*

$$Chow(k) \to DM^{eff}(k):$$

3.4 Projective bundle formula, etc

Let $X \in Sm=k$ and V a vector bundle on X. Let $\mathbb{P}(V)$ be the associated projective bundle. Then there exists a canonical isomorphism

$$M(\mathbb{P}(V)) \cong \oplus_{i=0}^{d-1} X(i)[2i]$$

where $X(i)[2i] = X \otimes \mathbb{Z}(i)[2i]$.

The proof is by induction. We have $\mathcal{O}(1)$ a line bundle on $\mathbb{P}(V)$ which gives a class in $Pic(\mathbb{P}(V))$ which gives a morphism $\mathbb{P}(V) \to \mathbb{Z}(1)[2]$ and so a morphism

$$\mathbb{P}(V) \to \mathbb{P}(V)^{\otimes i} \to \mathbb{Z}(i)[2i]$$

3.5 Motive of a blow-up

Let Z be a regular closed subscheme of a smooth scheme X.

Theorem 29. *There is a canonical isomorphism*

$$Bl_Z(X) \cong X \oplus \left(\oplus_{i=1}^{c-1} Z(i)[2i] \right):$$

3.6 Gysin triangle

There is a distinguished triangle

$$M(X \backslash Z) \to M(X) \to M(Z)(c)[2c] \to M(X \backslash Z)[1]$$

4 J. Ayoub, Motivic complexes and motivic cohomology V

4.1 Motivic cohomology and Milnor K-theory

Recall that we have defined $\mathcal{H}^{n,n} = H_0(C_\bullet^{\mathbb{A}^1}\mathbb{Z}_{tr}((\mathbb{G}_m; 1))^{\wedge n})$ = the first non-zero homology sheaf of $\mathbb{Z}(n)$. In particular, $H^{0,0} = \mathbb{Z}$.

Lemma 30.

$$\mathcal{H}^{1,1} = \mathcal{O}^*$$

Proof. There is a diagram $\mathbb{G}_m \to \mathbb{P}^1 \leftarrow \{0; \infty\}$. We have also $\mathcal{H}^{1,1}(K) = \frac{Cor^{\mathbb{A}^1}(K,\mathbb{G}_m)}{Cor^{\mathbb{A}^1}(K,1)} \cong Pic(\mathbb{P}^1; \{0; 1\})$, the relative Picard group.

$$0 \to \frac{k^* \oplus k^*}{k^*} \to Pic(\mathbb{P}^1; \{0; 1\}) \to Pic(\mathbb{P}^1)$$

and so $Pic(\mathbb{P}^1; \{0; 1\}) = k^* \oplus \mathbb{Z}$. Fact: $\mathcal{H}^{1,1}(k) = k^*$. There is a morphism $k^* \to H_0 C_\bullet \mathbb{Z}_{tr}(\mathbb{G}_m; 1)(k)$ which sends a to $[a]$ but we need to check $[ab] = [a] + [b]$. We will construct an $\mathbb{A}^1$-homotopy. We will find a $\quad \in Cor(\mathbb{A}^1; \mathbb{G}_m)$ with $\ \circ i_0 - \ \circ i_1 = [ab] - [a] - [b]$. Consider the correspondence $Z \subset \mathbb{A}^1 \times \mathbb{G}_m$ defined by

$$(q^2 - (t(1 + ab) + (1 - t)(a + b))q + ab) = 0$$

where $(t; q)$ are the coordinates of $\mathbb{A}^1 \times \mathbb{G}_m$ and remember that the cycle associated to 1 in $\mathbb{G}_m$ is already zero. □

Theorem 31.

$$H^{n,n}_M(k) \cong K^M_n(k)$$

Proof. Note that this is true for $n = 0; 1$. The idea of the proof is to define maps in both directions and show that their compositions are the identities.

Step 1: $H^{n,n}_M \to K^M_n(k)$. A typical element in $H^{n,n}_M(k)$ is a class of zero cycles in $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. Our map $Corr(k; (\mathbb{G}_m)^n)$ takes a point x to $N_{k(x)/k}(x_1; \ldots; x_n)$. We need to check the following. If we have a correspondence $\mathbb{A}^1 \to \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ then composition with 0 and 1 gives the same element of $K^M_n(k)$. A correspondence comes from a diagram

$$\begin{array}{ccc} C & \longrightarrow & \\ \downarrow & & \searrow \\ \mathbb{A}^1 & & \mathbb{G}_m \times \cdots \times \mathbb{G}_m \end{array}$$

where C is a smooth irreducible curve.

$$p^{-1}(0) = \{x_{01}; \ldots; x_{0n}\}$$
$$p^{-1}(1) = \{x_{11}; \ldots; x_{1n}\}$$

and we want to show

$$\sum_{i=1}^{n} N_{k(x_{0i})/k}\{f_1(x_{01}; \ldots; f_n(x_{0n})\} = \sum_{i=1}^{n} N_{k(x_{1i})/k}\{f_1(x_{11}; \ldots; f_n(x_{1n})\}$$

Consider $K^M_{n+1}(k(C)) \ni \{\frac{t}{t-1}; f_1; \ldots; f_n\}$. If $y \in \overline{C}$ a closed point, $\partial_y\{\frac{t}{t-1}; f_1; \ldots; f_n\}$. Everything then follows form the Weilreciprocity formula.

Step 2: $K^M_n(k) \to H^{n,n}_M$. The symbol $a_1 \otimes \cdots \otimes a_n$ will get sent to $[a_1; \ldots; a_n]$. We need to check that in $H^{2,2}(k)$ we have the relation $[a; 1 - a] = 0$. To show this we define a correspondence $\mathbb{A}^1 \to \mathbb{G}_m \backslash 1 \overset{(x,1-x)}{\to} \mathbb{G}_m \times \mathbb{G}_m$. Let $q; t$ be the coordinates for $\mathbb{G}_m; \mathbb{A}^1$ as before and we take the correspondence defined by

$$q^3 - t(a^3 + 1)q^2 + t(a^3 + 1)q - a^3 = 0$$

Step 3: It is then shown that the composition $K^M_n(k) \to H^{n,n}_M \to K^M_n(k)$ is the identity and that the first morphism is surjective. □

4.2 Étale motives

What happens if we use the étale topology instead of Nisnevich? We get

$$DM^{eff,et}(k) = D(PSh_{tr}) \text{=} \left\{ \begin{array}{c} \text{homotopy} \\ \text{Et top} \end{array} \right\}$$

$X; \mathbb{Z}_{tr}(X); M^{et}(X)$ = étale motive of X. This is isomorphic to $C_\bullet^{\mathbb{A}^1}\mathbb{Z}_{tr}(X)[\frac{1}{p}]$. The characteristic gets inverted automatically due to the short exact sequence $0 \to \mathbb{Z}\text{=}p \to \mathcal{O} \xrightarrow{p} \mathcal{O} \to 0$.

Lichtenbaum or étale motivic cohomology is defined as

$$\begin{aligned} H_L^{p,q} &= \hom_{DM^{eff,et}(k)}(X; \mathbb{Z}(q)[p]) \\ &= H_{et}^p(X; \mathbb{Z}(q)) \end{aligned}$$

Theorem 32.

$$H_M^{p,q}(X) \otimes \mathbb{Q} \cong H_L^{p,q}(X) \otimes \mathbb{Q}$$

Theorem 33.

$$H_L^{p,q}(X; \mathbb{Z}\text{=`}) \cong \textit{ etale cohomology of } X$$

The first theorem is not so hard. There is an equivalence of categories between $DM^{eff}(k; \mathbb{Q}) \cong DM^{eff,et}(k; \mathbb{Q})$. This comes from the equivalence $Shv_{Nis}^{tr}(Sm\text{=}k; \mathbb{Q}) \cong Shv_{et}^{tr}(Sm\text{=}k; \mathbb{Q})$. For example, if we take a presheaf with transfers then $F_{et} = 0$ implies that $F_{Nis} = 0$. To see this take a henselien ring S and we want to show that $F(S) = 0$, where we know that $F(\overline{S}) = 0$ where $\overline{S}$ is the strict henselisation. For any element though, this means that there is a finite extension $T \to S$ on which the element vanishes, and so using traces it follows that the element is itself zero (because we are using rational coefficients).

4.3 Suslin rigidity

Theorem 34. *There is a canonical equivalence of monoidal triangulated categories*

$$DM^{eff,et}(k; \mathbb{Z}\text{=`}) \cong D(Et\text{=}k; \mathbb{Z}\text{=`})$$

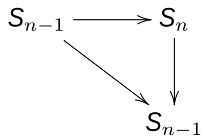
This is equivalent to say that if F is a homotopy invariant presheaf with transfers, with $\mathbb{Z}$=` coefficients, then the sheafification F_{et} is a locally constant sheaf. If k is seperably closed, then it is equivalent to say that F_{et} is constant.

Consider $X\text{=}S$ a smooth relative curve ($X; S$ affine), with an embedding $X \to \overline{X} \to S$ which is a good compactification, so $\overline{X}$ is a relative proper curve and $Z = \overline{X} \backslash X$ has an affine neighbourhood.

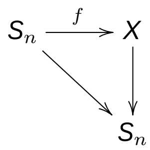
Theorem 35. *In this situation,* $Cor_S^{\mathbb{A}^1}(S; X) \cong Pic(\overline{X}; Z)$ *where the latter is the relative Picard group.*

Let F be a homotopy invariant presheaf with transfers of $\mathbb{Z}$=` modules. If S is a strictly henselien local ring then $F(S) \cong F(s)$ where s is the closed point.

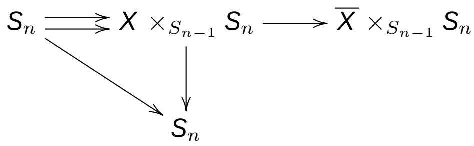
Assume that $S = (\mathbb{A}^n)^{sh}_0$



It suffices to show that $F(S^n) \cong F(S^{n-1})$. For this it suffices to show that id_{S^n} and $S^n \to S^{n-1} \to S^n$ are $\mathbb{A}^1$-homotopic. Find an relative with compactification



We try to show that $f \sim f \circ p \circ$ assuming that X has a good compactification.



Two elements of $Pic(\overline{X} \times_{S_{n-1}} S_n; Z \times_{S_{n-1}} S_n)$=`. When we pullback along the closed point $s \in S_n$ we find that the two elements of $Pic(\overline{X} \times_{S_{n-1}} s; Z \times_{S_{n-1}} s)$=` are equal. To finish the proof we just need to know that the specialisation map between relative picard groups is an injection.

Theorem 36.

$$Pic(\overline{X}; Z)=` \to Pic(\overline{X} \times_S s; Z \times_S s)=`$$

is an isomorphism.

$$Pic(\overline{X}; Z) \to H^2_{et}(\overline{X}; j_! \ _\ell)$$

and the latter is isomorphic to $H^2_{et}(\overline{X}; j_! \ _\ell|_{\overline{X}})$.

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