Motives and Milnor conjecture

notes by S. Kelly:

Day VII

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1 A. Asok/ A. S. Merkurjev - Proof of the Milnor conjecture

Recall that we constructed a group $H^{n+1,n}(\check{C}(X), \mathbb{Z}_{(\ell)})$ that controls the kernel of the map $H_{et}^{w+1,w}(E, \mathbb{Z}_{(\ell)}) \to H_{et}^{w+1,w}(E(X_\alpha), \mathbb{Z}_{(\ell)})$ that we want to show is injective.

Remark 1.

1. From now on, $\ell = 2$.
2. Reduced motivic cohomology will be denoted $\tilde{H}^{*,ast}(X)$.
3. If no ceofficients are given, it is assumed they are $\mathbb{Z}/2$.

Recall that we have operations

$$Q_i : \tilde{H}^{p,q}(\mathcal{X}) \to \tilde{H}^{p+2^{i+1}-1,q+2^i-1}(\mathcal{X})$$

1. $Q_i^2 = 0$
2. $Q_0 = B$ the Bockstein morphism $\tilde{H}^{*,*}(\mathcal{X}) \to \tilde{H}^{*+1,*}(\mathcal{X})$ associated to $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$ (Consider the connecting homomorphism $\tilde{\beta} : \tilde{H}^{*,*}(\mathcal{X}) \to \tilde{H}^{*+1,*}(\mathcal{X}, \mathbb{Z})$ associated with $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$).
3. $Q_iQ_j = Q_jQ_i$.

Definition (Motivic Margolis homology)**.**

$$\widetilde{MH}_i^{p,q} = ker(Q_i)/im(Q_i)$$

The goal of today is to prove the vanishing of some Margolis homology groups.

For any smooth scheme X, we can adjoin a disjoint base point and we get a canonical morphism $\check{C}(X)_+ \to S_F^0 \to \tilde{C}(X)$ (where S^0 is two disjoint points). Recall that if X has an F point, then $\tilde{C}(X)$ is contractible.

1.1 Warm-up

Suppose X is a smooth quadric. While X may be empty, X certainly has a rational point after passing to a degree 2 extension.

Lemma 2.

$$\widetilde{MH}_0^{*,*}(\check{C}(X)) = 0$$

Proof. If X has a point then it is clear. Assume $X(F) = \varnothing$. Let L/F be a quadratic extension such that X has an L point. Then we have a map $X_L \to X$ which induces a map $\tilde{C}(X_L) \to \tilde{C}(X)$.

$$\tilde{H}^{*,*}(\tilde{C}(X), \mathbb{Z}) \longrightarrow \tilde{H}^{*,*}(\tilde{C}(X_L), \mathbb{Z})$$

and composition of the two maps is multiplication by 2. The reduced homology is zero and so two times any element on the left is zero. Thus, this gives what we want:

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$$

induces a long exact sequence which splits into short exact sequences (because multiplication by two is zero), and then a quick diagram chase finishes the proof. □

1.2 Interlude

There exists a characteristic class s_d associated to the Newton symmetric polynomial $\sum_i t_i^d$. Firstly, this is a natural transformation and so behaves well on exact sequences, and secondly, $s_d(\mathcal{L}) = c_1(\mathcal{L})^d$ for a line bundle $\mathcal{L}$.

Definition. A ν_n variety (at 2) is a smooth proper variety X such that

1. $\dim X = 2^n - 1$
2. *deg* $s_{2^n-1}(X) \not\equiv 0$ mod 4 (where $s_{2^n-1}(X)$ is s_{2^n-1} applied to the tangent bundle of X).

A ν_n point of a variety Y is a morphism from a ν_n variety to Y.

Lemma 3. *If X is a smooth quadric in $\mathbb{P}^n$ then deg$(s_{n-1}(X)) = 2(n+1-2^{n-1})$.*

Idea of proof. Let $i : X \to \mathbb{P}^n$ be the inclusion. We have short exact sequences

$$0 \to T_X \to i^*T_{\mathbb{P}^n} \to i^*\mathcal{O}(2) \to 0$$

which describes the tangent bundle of X and

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

□

Corollary 4. *If X is a smooth quadric in $\mathbb{P}^{2^n}$ then deg $s_{2^n-1}(X) \equiv 2$ mod 4, i.e. X is a ν_n variety.*

Since we can find smooth subquadrics of X of dimension $2^i - 1$, then X has ν_i points for all $0 \leq i \leq n$.

Theorem 5. *Suppose X is a smooth projective variety that has a ν_i point, then the $\widetilde{MH}_i^{*,*}(\tilde{C}(X)) = 0$.*

Corollary 6. *If X is a smooth quadric of dimension $2^n - 1$ then $\widetilde{MH}_i^{*,*}(\tilde{C}(X)) = 0$ for $0 \leq i \leq n$.*

Proof. The idea is to construct a contracting homotopy.

Step 1. Construct a map $\tilde{H}^{p+2^{i+1}-1,q+2^i-1} \to \tilde{H}^{p,q}(\check{C}(X))$. To construct this we will use duality. Recall that if X is a smooth projective variety of dimension d there exists an integer n, a vector bundle V on X of rank n, and a morphism $f_V : T^{n+d} \to Th_X(V)$ such that

1. $[V \oplus T_X] = 0$ in $K_0(X)$
2. the map $H^{2d,d}(X) \to \mathbb{Z}$ induced by f_V via the Thom isomorphism and suspension isomorphism coincides with the usual degree map.

Consider the cone of the map $f_V : T^{n+d} \to Th_X(V)$. Henceforth, X is our smooth projective variety, and Y our ν_i variety and we have a morphism $Y \to X$. Let $d = 2^i - 1$, the dimension of Y. There is a Thom class $t_V \in \tilde{H}^{2n,n}(Th_Y(V), \mathbb{Z})$ and it can be lifted to a unique class α in $\tilde{H}^{2n,n}(cone(f_V))$. Cupping with α induces a map $\tilde{H}^{p,q}(\tilde{C}(X) \wedge cone(f_V), \mathbb{Z}) \to \tilde{H}^{p+2n,q+n}(\tilde{C}(X) \wedge cone(f_V), \mathbb{Z})$. If we smash the exact sequence

$$T^{n+d} \to Th_Y(V) \to Cone(f_V) \to \Sigma_s^1 T^{n+d}$$

with $\tilde{C}(X)$ we get a cofiber sequence

$$T^{n+d} \wedge \tilde{C}(X) \to Th_Y(V) \wedge \tilde{C}(X) \to Cone(f_V) \wedge \tilde{C}(X) \to \Sigma_s^1 T^{n+d} \wedge \tilde{C}(X)$$

and we get a map $\tilde{H}^{*,*}(\Sigma_s^1 T^{\wedge n+d} \wedge \tilde{C}(X)) \to \tilde{H}^{*,*}(Cone(f_V) \wedge \tilde{C}(X))$.

Lemma 7. *$Th_Y(V)\tilde{C}(X)$ is contractible (thus $Cone(f_V) \wedge \tilde{C}(X) \to \Sigma_s^1 T^{n+d} \wedge \tilde{C}(X)$ is a weak equivalence).*

Proof. This should just be writing down definitions as well as a huge diagram and tracking the maps that appear. □

Given the lemma, we get composite maps

$$\tilde{H}^{p,q}(\tilde{C}(X), \mathbb{Z}) \to \tilde{H}^{p+2n,q+n}(Cone(f_V) \wedge \tilde{C}(X)) \cong \tilde{H}^{p+2n,q+n}(\Sigma_s^1 T^{n+d} \wedge \tilde{C}(X))$$

giving

$$\phi : \tilde{H}^{p,q}(\tilde{C}(X), \mathbb{Z}) \to \tilde{H}^{p-2d-1,q-d}(\tilde{C}(X), \mathbb{Z})$$

Step 2. Claim: If x is in $\tilde{H}^{p,q}(\tilde{C}(X))$ then $x \equiv Q_i\phi(x) + \phi Q_i(x)$. Let γ be the image of the canonical element of $\tilde{H}^{p+2d+1,q+d}(\Sigma_s^1 T^{\wedge n+d})$ in $\tilde{H}^{p+2d+1,q+d}(cone(f_V))$. It suffices to prove that $\gamma \wedge x = \alpha \wedge Q_i(x) + Q_i(\alpha \wedge x)$. We compute $Q_i(\alpha \wedge x) =$

$Q_i\alpha \wedge x + \alpha \wedge Q_i(x) +$ some other terms of the form $\prod_{j<i} Q : j(\alpha) \wedge x$. Another fact is that Q_i kills Thom classes of vector bundles. One deduces that for $j < i$ we have $Q_j(\alpha) = 0$ and so we get

$$Q_i(\alpha) \wedge x = \alpha \wedge Q_i(x) + Q_i(\alpha \wedge x)$$

We have to show that $Q_i(\alpha) = \gamma$.

So far we know $Q_i(\alpha) \wedge x = \alpha \wedge Q_i(x) + Q_i(\alpha \wedge x)$ and we want to show that $Q_i(\alpha) = \gamma$.

We have $Q_i = [\beta, q_i]$, and $Q_i(\alpha) = \beta q_i(\alpha) + q_i\beta(\alpha)$ where $q_i\beta(\alpha)$ is actually zero. It suffices to show that $q_i(\alpha)$ cannot be lefted to a class in $\mathbb{Z}/4$ coefficients. Also, we have $q_i(t_V))s_d(V)t_V$. When Y is a smooth projective quadric, Y has no points over extensions of odd degrees, and this is equivalent to a statement about $H^{2n,n}(Y) \overset{deg}{\to} \mathbb{Z} \to \mathbb{Z}/2$. Suppose $q_i(\alpha)$ can be lifted to a class z in motivic cohomology with $\mathbb{Z}/4$ coefficients such that $f_V^*(z) = 0$. The Y has a point over an extension of odd degree X which contradicts our assumption that $deg\ s_s(Y) \equiv 2 \bmod 4$. □

Recall that the proof was divided into several steps.

Step 1. $MH90(n)$ for fields F, 2-special with $k_n(F) = 0$.

Step 2. Reduction to the injectivity of a certain map $H_{et}^{w+1,w}(F, \mathbb{Z}_{(2)}) \to H_{et}^{w+1,w}(F(X_\alpha), \mathbb{Z}_{(2)})$ where X_α is the quadric associated to $q_\alpha = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$ so $\dim X_\alpha = 2^{n-1} - 1$.

Step 3. Reduction to the triviality of $H^{n+1,n}(\mathcal{X}_\alpha, \mathbb{Z})$ (where $\mathcal{X} = \check{C}(X_\alpha)$).

Step 4. Reduction to the triviality of $H^{2^n-1,2}(\mathcal{X}_\alpha, \mathbb{Z})$.

Step 5. Proof that $H^{2^n-1,2^{n-1}}(\mathcal{X}_\alpha, \mathbb{Z})$ is trivial.

Voevodsky had three main ideas, each of which would have been lifetime acheivements for a mathematician.

1. Motivic cohomology, which fits very nicely with Milnor K-theory and étale cohomology.

2. To use $\check{C}(X)$, which simplifies the cohomology greatly, but retains the essential information.

3. Steenrod operations.

Define $\tilde{C}(X_\alpha) = \tilde{\mathcal{X}}_\alpha$. We want to compare the motivic cohomology of $\tilde{\mathcal{X}}_\alpha$ with that of X_α.

$$H^{p-1,q}(F, A) \to H^{p-1,q}(\mathcal{X}_\alpha, A) \to \tilde{H}^{p,q}(\tilde{\mathcal{X}}_\alpha, A) \to H^{p,q}(F, A) \to H^{p,q}(\mathcal{X}_\alpha, A)$$

Lemma 8.

1. $H^{p-1,q}(\mathcal{X}_\alpha, \mathbb{Z}) \cong \tilde{H}^{p,q}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z})$.

2. $\tilde{H}^{p,q}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$ *if* $p-1 \leq q < n$.

$$BL(q) \qquad i \leq j < n \qquad H^{i,j}(\mathcal{X}_\alpha, \mathbb{Z}/2) \cong H^{i,j}_{et}(\mathcal{X}_\alpha, \mathbb{Z}/2) = H^{i,j}_{et}(F, \mathbb{Z}/2) = H^{i,j}(F, \mathbb{Z}/2)$$

Take the standard short exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$, we get

$$\begin{array}{ccccc} \tilde{H}^{p,q}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}) & \xrightarrow{inj} & \tilde{H}^{p,q}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) & \longrightarrow & \tilde{H}^{p+1,q}(\tilde{X}_\alpha, \mathbb{Z}) \\ & & & \searrow^{\beta} & \downarrow inj \\ & & & & \tilde{H}^{p+1,q}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) \end{array}$$

$u \in \tilde{H}^{p,q}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2)$ is integral if and only if $\beta u = 0$. $\beta Q_i(u) = Q_i \beta(u) = 0$ implies that $Q_i(u)$ is integral.

Proposition 9. $Q_{n-2} \circ \cdots \circ Q_2 \circ Q_1 : \tilde{H}^{n+2,n}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) \to \tilde{H}^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2)$ *is injective.*

Proof. Choose u such that $Q_{n-2} \circ \cdots \circ Q_2 \circ Q_1(u)$ and prove by descending induction that $Q_i \circ Q_{i-1} \cdots \circ Q_2 \circ Q_1(u) = 0$. Suppose that we have $Q_i \circ Q_{i-1} \cdots \circ Q_2 \circ Q_1(u) = 0$ and we want to show that $Q_{i-1} \cdots \circ Q_2 \circ Q_1(u) = 0$. We know that $MH_i(\tilde{X}_\alpha) = 0$ and so $Q_{i-1} \dots Q_1(u) = Q_i(v)$ for $v \in \tilde{H}^{n-i,n-i}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2)$. By the lemma this group is trivial. □

$$\begin{array}{ccc} \tilde{H}^{n+2,n}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) & \longrightarrow & \tilde{H}^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) \\ \uparrow & & \uparrow \\ \tilde{H}^{n+2,n}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}) & \longrightarrow & \tilde{H}^{2^n,2^{n-1}}(\tilde{\mathcal{X}}_\alpha, \mathbb{Z}) \\ \| & & \| \\ H^{n+1,n}(\mathcal{X}_\alpha, \mathbb{Z}) & \longrightarrow & H^{2^n-1,2^{n-1}}(\mathcal{X}_\alpha, \mathbb{Z}) \end{array}$$

It remains to show that $H^{2^n-1,2^{n-1}}(\mathcal{X}_\alpha, \mathbb{Z})$ is trivial. If we write $d = 2^{n-1} - 1 = dim\ X_\alpha$ this group becomes

$$H^{2d+1,d+1}(\mathcal{X}_\alpha, \mathbb{Z}).$$

We have an exact triangle

$$\mathcal{X}_\alpha(d)[2d] \to M_\alpha \to \mathcal{X}_\alpha(d)[2d+1]$$

where M_α is the Rost motive, which gives an exact sequence

$$H^{0,1}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2d+1,d+1}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2d+1,d+1}(M_\alpha, \mathbb{Z}) \xrightarrow{t} H^{1,1}(\mathcal{X}_\alpha, \mathbb{Z})$$

We can compute

$$H^{i,1}(\mathcal{X}_\alpha,\mathbb{Z}) \cong H^{i,1}_{et}(\mathcal{X}_\alpha,\mathbb{Z}) = H^{i,1}_{et}(F,\mathbb{Z}) = \left\{ \begin{array}{ll} 0 & i=0 \\ F^* & i=0 \end{array} \right.$$

The Rost motive M_α is a direct summand of $M(X_\alpha)$ and of $M(Q_\alpha)$ where $Q_\alpha = Q(\langle\langle a_1,\ldots,a_n\rangle\rangle) \supset X_\alpha$. We would like to understand $H^{2d+1,d+1}(X,\mathbb{Z})$ and we know that $d = \dim X$. This group is easier to deal with because it is a border case. Recall that $H^{i,j}(X,\mathbb{Z}) = 0$ if $i > j+d$. So we can use the Gerston complex to get

$$H^{2d+m,d+m}(X,\mathbb{Z}) = coker\Big(\amalg_{x\in X_{(1)}} K^M_{m+1}F(x)) \to \amalg_{x\in X_{(0)}} K^M_m F(x))\Big) = A_0(X,K_m)$$

The norm map sends the latter to $K^M_m(F)$. So $H^{2d+1,d+1}(M_\alpha,\mathbb{Z})$ is a direct summand of $H^{2d+1,d+1}(X_\alpha,\mathbb{Z}) = A_0(X,K_1)$ and t is the restriction of the norm map. So our exact sequence becomes

$$0 \to H^{2d+1,d+1}(\mathcal{X}_\alpha,\mathbb{Z}) \to \underbrace{H^{2d+1,d+1}(M_\alpha,\mathbb{Z})}_{\subset A_0(X,K_1)} \xrightarrow{t} F^*$$

We will show that the norm map is injective.

If $X_\alpha = C$ a conic curve (this is the case $n=2$) we need to show that the sequence $K_2(F(C)) \to \amalg_{x\in C_{(0)}} K_1(F(x)) \xrightarrow{N} F^*$ is exact. Milnor's theorem if C is split.

If X_α is isotropic then $M(X_\alpha) = \mathbb{Z}\oplus M(X'_\alpha)(1)[2]\oplus\mathbb{Z}(d)[2d]$ and so $A_0(X_\alpha,K)$ is a direct sum of three groups, the first two of which are zero, and the last is isomorphic to F^*.

Claim: The image of the norm map is $D(\phi_\alpha)$ where $\phi_\alpha = \langle\langle a_1,\ldots,a_n\rangle\rangle$. Proof: Let $x \in Q_\alpha$ be a closed point.

$$_{N_{F(x)/F}(F(x)^*) = D((\phi_\alpha)_{F(x)})} \overset{?}{\subset} D(\phi_\alpha)$$

$(\phi_\alpha)_{F(x)}$ isotorpic, Knebusch's norm principle.

Take $\phi_\alpha(v)$. Fix a vector v_0 such that $\phi_\alpha(v_0) = 1$, then we can write any vector as $v = bv_0 + w$ with $w \in v_0^\perp$. We have $\phi_\alpha(v) = \phi_\alpha(bv_0+w) = b^2 - a = N(b+\sqrt{a})$ where $a = -\phi_\alpha(w)$. Set $W = Span(v_0,w)$. We have $Q_\alpha \supset Q(\phi_\alpha|_w) = Spec(F(\sqrt{a})) = Q(\langle 1,-a\rangle)$. So $\phi_\alpha(v) \in N_{F(x)}(b+\sqrt{a})$.